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Borel-Laplace transformations and invariant curves for the Hénon maps

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Let a and b be complex numbers with $a \neq 0$, and define $f = f_{a,b} : \mathbb{C}^2 \to \mathbb{C}^2$ by

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 1 + y - ax^2 \\ bx \end{pmatrix},$$

which is called the *Hénon map.* If $b = 0$, the Hénon map $f = f_{a,0} : \mathbb{C}^2 \to \mathbb{C}^2$ is consistent with a quadratic function $\mathbb{C} \to \mathbb{C}$ defined by $x \mapsto 1 - ax^2$. Let $P = (x_f, y_f)$ be a fixed point of f, and let α be an eigenvalue of the derivative

$$Df_P = \begin{pmatrix} \lambda & 1 \\ b & 0 \end{pmatrix}, \quad \lambda = -2ax_f.$$

Suppose $\alpha \neq 0$. For $n \geq 1$ we let

$$\begin{aligned} D_n &= \alpha^n - \lambda - b\alpha^{-n} \\ &= \alpha^{-n}(\alpha^n - \alpha_1)(\alpha^n - \alpha_2), \end{aligned}$$

where α_1 ad α_2 are eigenvalues of Df_P. It is evident that $D_1 = 0$. Throughout this paper we assume the following.

Basic assumption. *For all $n \geq 2$, $D_n \neq 0$.*

By a result of Poincaré [P] it follows that if $|\alpha| \neq 1$ and $\mathbf{v}_\alpha$ is an eigenvector of Df_P for α, then there is uniquely an analytic map $\phi : \mathbb{C} \to \mathbb{C}^2$ such that $\phi(0) = P$, $\phi'(0) = \mathbf{v}_\alpha$, and $f \circ \phi(t) = \phi(\alpha t)$ for all $t \in \mathbb{C}$. We call the analytic conjugacy (or semi-conjugacy) ϕ the *Poincaré map.* Note that if $b \neq 0$ then ϕ is univalent. When P is hyperbolic, i.e. $0 < |\alpha_1| < 1 < |\alpha_2|$, if $\alpha = \alpha_1$ then $\phi(\mathbb{C})$ is consistent with the *stable manifold* of P and if $\alpha = \alpha_2$ then $\phi(\mathbb{C})$ is the *unstable manifold* of P.

In the case where $|\alpha| = 1$, let $\alpha = e^{2\pi i\theta}, \theta \in \mathbb{R}$. From a result of Siegel [S] we have that if $b = 0$ and if there are constants $c, d > 0$ such that

$$|\theta - \frac{p}{q}| > \frac{c}{q^d} \quad (\forall p, q \in \mathbb{Z},\ q \geq 1)$$

then there is uniquely a univalent analytic map $\phi : D_1 = \{z \in \mathbb{C} \mid |z| \leq 1\} \to \mathbb{C}^2$, the *Poincaré map* by definition, such that $\phi(D_1) \subset \mathbb{C} \times \{0\}$, $\phi(0) = P$, $\phi'(0) = \mathbf{v}_\alpha$, and

$f \circ \phi(t) = \phi(\alpha t)$ for all $t \in D_1$. We remark that the set of θ's satisfying the condition above is of full measure ([S]).

Let $\frac{p_n}{q_n}$ $(n = 1, 2, \cdots)$ be the n-th convergent of θ. We say that θ is a *Bruno number* if

$$\sum_{n=1}^{\infty} \frac{\log q_{n+1}}{q_n} < +\infty.$$

It can be checked that if θ satisfies the Siegel condition above then it is a Bruno number. By results of Bruno [B] and Yoccoz [Y] we have that, in the case where $b = 0$, θ is a Bruno number if and only if f is linearized by the Poincaré map at P, i.e. there is a univalent analytic map $\phi : D_1 \to \mathbb{C}^2$ such that $\phi(D_1) \subset \mathbb{C} \times \{0\}$, $\phi(0) = P$, $\phi'(0) = \mathbf{v}_\alpha$, and $f \circ \phi(t) = \phi(\alpha t)$ for all $t \in D_1$.

Let $A_1 \neq 0$, and define

$$A_n = \frac{A_1 A_{n-1} + A_2 A_{n-2} + \cdots + A_{n-1} A_1}{D_n} \qquad (n = 2, 3, \cdots).$$

Then it can be also checked that, in the case of $b = 0$, θ is a Bruno number if and only if the following condition is satisfied. See §1.

Condition (*). *There is* $M > 0$ *such that* $|A_n| \leq \mathrm{e}^{nM}$ *for all* $n \geq 2$.

In comparison with the linearization stated above, we consider the functional equation of form

$$f \circ \varphi(t) = \varphi(t + 1). \tag{0.1}$$

It is easy to see that if ϕ is the Poincaré map and we let $\varphi(t) = \phi(\alpha^t)$ then φ satisfies the equation (0.1). The purpose of this paper is to cunstruct a solution to the equation (0.1) by the method of Borel-Laplace transform which is developed by Écalle [E] and so on. In the case where $|\alpha| \neq 1$, it will result in getting the solutions φ which are different from the Poincaré maps ϕ in the sense that there are no analytic maps p such that $\varphi = \phi \circ p$.

Let $B_1 = 1$, and define

$$B_n = \frac{B_1 B_{n-1} + B_2 B_{n-2} + \cdots + B_{n_1} B_1}{n! D_n} \qquad (n = 2, 3, \cdots).$$

To perform Laplace transform, the following condition will be needed.

Condition (**). *There is* $M > 0$ *such that* $|n! B_n| \leq \mathrm{e}^{nM}$ *for all* $n \geq 2$.

In the case where $0 < |\alpha| < 1$, we have that $|n! B_n| \to 0$ as $n \to \infty$, and so Condition (**) is satisfied. In this case we will obtain an analytic map $\varphi : \mathbb{C} \to \mathbb{C}^2$, a solution to (0.1), with the property that $\varphi(t) \to P$ as $t \to \mathrm{e}^{i\Theta}\infty$ if $\mathrm{e}^{i\Theta} \neq -1$. Also, for the case of $|\alpha| > 1$, the similar result will be obtained.

For the case of $|\alpha| = 1$, the following theorem will be proved.

Theorem 1. *Suppose $|\alpha| = 1$ and $\alpha_1 \neq \alpha_2$. Then, under Condition (**), for $\varepsilon > 0$ there is an analytic map $\varphi : H = \{z \in \mathbb{C} \mid Im\ z > R\} \to \mathbb{C}^2$, with $\varphi'(t) \neq 0$ for all $t \in H$, such that $f \circ \varphi(t) = \varphi(t+1)$ for all $t \in H$, if $\operatorname{Im} t \to +\infty$ then $\varphi(t) \to P$ and $\varphi'(t)$ converge to an eigenvector for α, and $\varphi(H)$ is contained in the ε-neighborhood of P.*

Question 1. *Is there D, a complex 1-open disc, such that $\varphi(H) \subset D$ holds ?*

It is evident that if $b = 0$ then the answer to Question 1 is affirmative.

If $\alpha = \alpha_1$ is of modulus one and θ is a Bruno number and if $|\alpha_2| \neq 1$, then Condition (**) is satisfied (see Fact below).

In this case, the answer to Question 1 is affirmative and, by a result of Bruno [B], there is an f-invariant complex 1-open disc D such that $f_{|D} : D \to D$ is analytically conjugate to a rotation.

Question 2. *In the case above, is there $p : H \to D$, an analytic map, such that $\varphi = \phi \circ p$ holds ?*

Fact. *If $\alpha = \alpha_1$ is of modulus one and θ is a Bruno number and if $|\alpha_2| \neq 1$, then $n!B_n \to 0$ as $n \to \infty$.*

This is checked as follows. For $k \geq 1$ choose n_k such that $q_{n_k} \leq k < q_{n_k+1}$. Since θ is a Bruno number, obviously $\frac{\log q_{n_k+1}}{q_{n_k}} \approx 0$ if k is sufficiently large. Let k be sufficiently large, and take $M > 0$ such that for all ℓ with $1 \leq \ell \leq k$, $|\ell! B_\ell| \leq e^{M\ell}$. Then by Stirling's formula

$$|B_\ell| \leq \frac{1}{\ell!} e^{M\ell} \leq \mathrm{e}^{-N\ell \log \ell + \ell + M\ell},$$

where $N > 0$ is a constant, and hence

$$\begin{aligned}|B_1||B_k| + \cdots + |B_k||B_1| &\leq \mathrm{e}^{-Nk\log k + k + Mk} + \cdots + \mathrm{e}^{-Nk\log k + k + Mk} \\ &\leq \mathrm{e}^{-Nk \log \frac{k}{2} + k + 1 + \log(k+1) + M(k+1)}.\end{aligned}$$

Therefore, we have

$$|(k+1)!B_{k+1}| \leq \frac{1}{D_{k+1}} \mathrm{e}^{-Nk\log\frac{k}{2} + k + 1 + \log(k+1) + M(k+1)},$$

and since $\frac{1}{2q_{n_k} q_{n_k+1}} \leq |\frac{p_{n_k}}{q_{n_k}} - \theta| \leq \frac{1}{q_{n_k} q_{n_k+1}}$, it follows that the right side is

$$\begin{aligned}&\leq \mathrm{e}^{\log q_{n_k+1} - Nk\log\frac{k}{2} + k + 1 + \log(k+1) + M(k+1)} \\ (0.2) \qquad &\leq \mathrm{e}^{\{\frac{\log q_{n_k+1}}{q_{n_k}} - N\frac{k}{k+1}\log\frac{k}{2} + 1 + \frac{\log(k+1)}{k+1} + M\}(k+1)} \\ &\leq \mathrm{e}^{M(k+1)},\end{aligned}$$

which implies that $|n!B_n| \leq \mathrm{e}^{nM}$ for all $n \geq 2$ and, moreover, by (0.2) we have that $n!B_n \to 0$ as $n \to 0$. Therefore, Fact is obtained.

In this paper we only discuss the case of fixed points of the Hénon maps. The authors hope that the results in this paper is extended to the case of periodic points.

§1 Poincaré maps

As before, let $\alpha \neq 0$ be one of eigenvalues of the derivative Df_P of the Hénon map f at a fixed point $P = (x_f, y_f)$, and let $\mathbf{v}_\alpha = (\tilde{a}_1, \tilde{b}_1)$ be an eigenvalue for α. It follows that $\tilde{a}_1 \neq 0$.

Let $\phi(t) = (x(t) + x_f, y(t) + y_f)$. Then

$$f \circ \phi(t) = f \begin{pmatrix} x(t) + x_f \\ y(t) + y_f \end{pmatrix} = \begin{pmatrix} y(t) + \lambda x(t) - a\{x(t)\}^2 + x_f \\ bx(t) + y_f \end{pmatrix}$$

and $\phi(\alpha t) = (x(\alpha t) + x_f, y(\alpha t) + y_f)$. Assuming $f \circ \phi(t) = \phi(\alpha t)$, we have

$$x(\alpha t) - \lambda x(t) - bx(\alpha^{-1}t) = -a\{x(t)\}^2. \tag{1.1}$$

Expand $x(t)$ in a formal power series

$$x(t) = \sum_{n=1}^{\infty} \tilde{a}_n t^n,$$

and substitute this into (1.1). Then, comparing coefficients of terms of t^n on both sides, we obtain the coefficients

$$\begin{aligned} \tilde{a}_2 &= -\frac{a\tilde{a}_1^2}{D_2}, \quad \tilde{a}_3 = -\frac{2a\tilde{a}_1\tilde{a}_2}{D_3}, \ \dots \\ \tilde{a}_n &= -\frac{a(\tilde{a}_1\tilde{a}_{n-1} + \tilde{a}_2\tilde{a}_{n-2} + \dots \tilde{a}_{n-2}\tilde{a}_2 + \tilde{a}_{n-1}\tilde{a}_1)}{D_n}, \ \dots . \end{aligned}$$

We remark that if a, b, α_1, α_2 and $\tilde{a}_1$ are real numbers, then so are all coefficients $\tilde{a}_n$'s.

Lemma 1.1. *If* $0 < |\alpha| < 1$, *then there exists* $C > 0$ *such that for all* $n \geq 1$,

$$|\tilde{a}_n| < C^n |\alpha|^{n \log n}. \tag{1.2}$$

Proof. Choose n_0 such that for all $n \geq n_0$,

$$\left|\frac{a(n-1)}{D_n}\right|^{\frac{1}{n}} |\alpha|^{-\log 2} < 1,$$

and take $C > 0$ such that

$$|\tilde{a}_n|^{\frac{1}{n}} \leq C|\alpha|^{\log n} \quad (1 \leq \forall n \leq n_0).$$

If (1.2) is true for $n_0 \leq i \leq n$, then

$$|\tilde{a}_{n+1}|^{\frac{1}{n+1}} \leq |\frac{a}{D_{n+1}}|^{\frac{1}{n+1}} (\sum_{i=1}^{n} |\tilde{a}_i||\tilde{a}_{n+1-i}|)^{\frac{1}{n+1}}$$
$$\leq |\frac{an}{D_{n+1}}|^{\frac{1}{n+1}} (\max_{1\leq i\leq hn} |\tilde{a}_i||\tilde{a}_{n+1-i}|)^{\frac{1}{n+1}},$$

and, choosing i_0 as $|\tilde{a}_{i_0}||\tilde{a}_{n+1-i_0}| = \max_{1\leq i\leq n} |\tilde{a}_i||\tilde{a}_{n+1-i}|$, we have that the right side is

$$\leq |\frac{an}{D_{n+1}}|^{\frac{1}{n+1}} (|\tilde{a}_{i_0}|^{\frac{1}{i_0}})^{\frac{i_0}{n+1}} (|\tilde{a}_{n+1-i_0}|^{\frac{1}{n+1-i_0}})^{\frac{n+1-i_0}{n+1}}$$
$$\leq |\frac{an}{D_{n+1}}|^{\frac{1}{n+1}} (C|\alpha|^{\log i_0})^{\frac{i_0}{n+1}} (C|\alpha|^{\log(n+1-i_0)})^{\frac{n+1-i_0}{n+1}}$$
$$\leq |\frac{an}{D_{n+1}}|^{\frac{1}{n+1}} C|\alpha|^{\frac{i_0}{n+1}\log i_0 + \frac{n+1-i_0}{n+1}\log(n+1-i_0)}$$
$$\leq |\frac{an}{D_{n+1}}|^{\frac{1}{n+1}} C|\alpha|^{\log\frac{n+1}{2}}$$
$$\leq C|\alpha|^{\log(n+1)}.$$

Therefore, (1.2) holds for all $n \geq 1$.

In the case where $0 < |\alpha| < 1$, by Lemma 1.1 we obtain that $x(t)$ is an entire function. Since $y(t) = bx(\alpha^{-1}t)$, it follows that $y(t)$ is also an entire function. Letting $A_n = -a\tilde{a}_n$ for all $n \geq 1$. we see tha Condition (*) is satisfied. If $|t| << 1$, then since $\phi'(0) = (x'(0), y'(0)) = (\tilde{a}_1, b\alpha^{-1}\tilde{a}_1) = \mathrm{v}_\alpha \neq (0,0)$, $t \mapsto \phi(t)$ is injective, and hence $\phi'(t) \neq (0,0)$. Let $b \neq 0$. Then f is a diffeomorphism. Since $\phi(\alpha^n t) = f^n \circ \phi(t)$ for all $n \geq 0$, it follows that $\phi : \mathbb{C} \to \mathbb{C}^2$ is injective. It is easy to see that

$$\phi'(\alpha^n t) = \frac{1}{\alpha^n} Df^n_{\phi(t)}\phi'(t),$$

and therefore $\phi'(t) \neq (0,0)$ for all $t \in \mathbb{C}$.

The above discussion also works for the case of $|\alpha| > 1$. Hence, we obtain the same results in the case where $|\alpha| > 1$,

Proposition 1.2. *Let $x(t)$ be as above and suppose $|\alpha| \neq 1$. Then $x(t)$ is an entire function and $\phi : \mathbb{C} \to \mathbb{C}^2$ defined by*

$$\phi(t) = (x(t) + x_f,\ bx(\alpha^{-1}t) + y_f) \tag{1.3}$$

is the Poincaré map. Coversely, any Poincaré map is of this form.

The followin proposition is easily obtained.

Proposition 1.3. *Let $x(t)$ be as above and suppose $|\alpha| = 1$. Then, under Condition (*), there is an f-invariant complex 1-open disc D, containing P, such that $f : D \to D$ is analytically conjugate to a rotation $t \mapsto \alpha t$ on D_1. In addition, $\phi : \mathbb{C} \to \mathbb{C}^2$ defined by (1.3) is the Poincaré map and, coversely, any Poincaré map is of this form.*

By results of Bruno [B] and Yoccoz [Y], we also obtain the following.

Proposition 1.4. *Let $b = 0$. Suppose α is of modulus one, and let $\alpha = e^{2\pi i\theta}, \theta \in \mathbb{R}$. Then θ is a Bruno number if and only if Condition (*) is satisfied.*

§2 Borel-Laplace transform

In this section we consider an f-invariant curve at $P = (x_f, y_f)$ parameterized by the complex variable $t \in \mathbb{C}$ as follows;

$$t \mapsto \begin{pmatrix} x(t) + x_f \\ y(t) + y_f \end{pmatrix} = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}$$

such that

$$f : \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} \mapsto \begin{pmatrix} X(t+1) \\ Y(t+1) \end{pmatrix} = \begin{pmatrix} 1 + Y(t) - aX(t)^2 \\ bX(t) \end{pmatrix}.$$

Then, the following difference equation of the second kind is obtained:

$$x(t+1) - \lambda x(t) - bx(t-1) = -a\{x(t)\}^2, \tag{2.1}$$

and $y(t) = bx(t-1)$. It is easy to see that a power series of form

$$x(t) = \sum_{n=0}^{\infty} \frac{a_n}{t^{n+1}}$$

is a formal solution to (2.1) if and only if $a = 1$ and $b = -1$, i.e. $\alpha_1 = \alpha_2 = 1$, which is an excluded case by Basic assumption and make all the difference from the discussion in §1. Note that, in this case, we have the power series

$$x(t) = \sum_{n=0}^{\infty} \frac{a_n}{t^{n+1}} = -\frac{6}{t^2} + \frac{15}{2t^4} - \frac{663}{40t^6} + \cdots,$$

and the difference equation (2.1) is related with the ordinary differential equation

$$\frac{d^2}{dt^2} x(t) = -\{x(t)\}^2$$

under the correspondance of $x(t+1) - x(t)$ to $\frac{\mathrm{d}}{\mathrm{d}t}$. Thus, the difference equation (2.1) discussed in this paper is far from the integrable systems except the case of $a = 1$ and $b = -1$, and the perturbation theory does not work.

To solve (2.1), we express $x(t)$ as a Laplace transformation of some Riemann surface X;

(2.2)
$$x(t) = \mathcal{L}[X](t) = \int_\gamma e^{-\zeta t} X(\zeta) d\zeta.$$

The contour γ is chosen later, depending on the positions and forms of branch points of X, such that

(1) if $Y(\zeta)$ is an entire function and is of exponential type, i.e. there are constants $C, M > 0$ such that $|Y(\zeta)| \leq C e^{M|\zeta|}$, then

$$\int_\gamma e^{-\zeta t} Y(\zeta) d\zeta = 0,$$

and

(2)
$$\{\int_\gamma e^{-\zeta t} X(\zeta) d\zeta\}^2 = \int_\gamma e^{-\zeta t} X * X(\zeta) d\zeta,$$

where $*$ denotes the convolution defined by

$$F * G = \int_0^\zeta F(\zeta - \zeta')G(\zeta')d\zeta'.$$

Then, from (2.1) it follows that

$$\begin{aligned}\int_\gamma e^{-\zeta t}(e^{-\zeta} - \lambda - be^{\zeta})X(\zeta)d\zeta &= -a\int_\gamma e^{-\zeta t} X * X(\zeta) d\zeta \\ &= \int_\gamma e^{-\zeta t}\{-aX * X(\zeta) + C(\zeta)\}d\zeta,\end{aligned}$$

where $C(\zeta)$ is is an entire function of exponential type. Letting

$$A(\zeta) = e^{-\zeta} - \lambda - be^{\zeta},$$

we see that if a Riemann surface X satisfies the integral equation

(2.3)
$$AX = -aX * X + C,$$

then a solution $x(t)$ to the difference equation (2.1) is obtained by the Laplace transformation (2.2).

If $X(\zeta)$ is a local solution to (2.3) in a neighborhood of the origin of $\mathbb{C}$, obviously $X * X(0) = 0$, and so $A(0)X(0) = C(0)$, from which it follows that

$$(1 - \lambda - b)X(0) = C(0).$$

After the construction of local solutions $X(\zeta)$ to (2.3), we will prove in §4 that $X(0) = (\alpha + b\alpha^{-1})/(2a)$, where $\alpha = \alpha_1, \alpha_2$ and $\alpha \neq 0$, which implies that the constant term $C(0)$ must coincide with

$$C(0) = \frac{(1-\lambda-b)(\alpha+b\alpha^{-1})}{2a}.$$

We remark that the entire function $C(\zeta)$ can be chosen as a constant function.

§3 Local solutions to the integral equation

In this section we construct local solutions $X(\zeta)$ to the integral equation (2.3) in a neighborhood of the origin of $\mathbb{C}$. To do this, we assume that $X(\zeta)$ is expressed as a Taylor series

$$X(\zeta) = a_0 + a_1\zeta + a_2\zeta^2 + \cdots$$

in a neighborhood of the origin, and define $\tilde{X}$ by

$$X(\zeta) = a_0 + \tilde{X}(\zeta). \tag{3.1}$$

Substitute (3.1) into (2.3). Then, we obtain

$$A\tilde{X} + 2aa_0 * \tilde{X} = W, \tag{3.2}$$

where

$$W = W_0 - a\tilde{X} * \tilde{X}, \quad W_0 = -aa_0^2\zeta - a_0A + C.$$

Let $A\tilde{X} = F$, and substitute this into (3.2). Then we have

$$F' + 2aa_0A^{-1}F = W', \tag{3.3}$$

where the prime denotes the derivative with respect to ζ. The solution to (3.3) is given by

$$F = F_0 \int_0^\zeta \frac{W'}{F_0} \mathrm{d}\zeta'$$

where

$$F_0 = \left(\frac{\mathrm{e}^{-\zeta} - \alpha_1}{\mathrm{e}^{-\zeta} - \alpha_2}\right)^\beta, \quad \beta = \frac{2aa_0}{\alpha_1 - \alpha_2} \tag{3.4}$$

is a solution to the following homogeneous equation:

$$F_0' + 2aa_0A^{-1}F_0 = 0.$$

If β is not an integer, the function F_0 in (3.4) is defined on the region

$$\mathcal{R}_0 = \mathbb{C} \setminus \{s\zeta \mid s \in [1, +\infty), \ \zeta = \zeta_k^+, \ \zeta_k^-, \ k \in \mathbb{Z}\},$$

where for $k \in \mathbb{Z}$

$$\zeta_k^+ = \rho_+ + (2k\pi + \theta_+)\mathrm{i}, \qquad \zeta_k^- = \rho_- + (2k\pi + \theta_-)\mathrm{i}$$

and

$$\rho_+ = -\log|\alpha_1|, \qquad \rho_- = -\log|\alpha_2|,$$
$$-\pi < \theta_+ = \arg\alpha_1 \le \pi, \qquad -\pi < \theta_- = \arg\alpha_2 \le \pi.$$

In the case where β is a positive integer, F_0 is a meromorphic function on $\mathbb{C}$ such that each ζ_k^- is a pole, and for the case of β a negative integer, each ζ_k^+ is a pole of F_0. It is clear that if $a_0 = 0$, F_0 is a constant function.

It turns out that the solution to (3.2) is

$$\tilde{X} = A^{-1}F_0 \int_0^\zeta \frac{W'}{F_0}\mathrm{d}\zeta'. \tag{3.5}$$

Hence, the solution $X(\zeta)$ to (2.3) can be sigular at the points: $\zeta = \zeta_k^+, \zeta_k^-$ $(k \in \mathbb{Z})$, where $A(\zeta) = 0$.

Now we let β be not an integer, and for $\epsilon > 0$ small, introduce the region

$$\mathcal{R}_\epsilon = \mathbb{C} \setminus \{s\zeta \mid s \in [1, +\infty),\ \zeta \in D(\zeta_k^+, \epsilon) \cup D(\zeta_k^-, \epsilon),\ k \in \mathbb{Z}\}$$

where $D(\zeta_k^\pm, \epsilon)$ denotes the open disk with radius ϵ centered at $\zeta_k^\pm$ respectively. To give an algorithm to construct the solution $\tilde{X}(\zeta)$ on $\mathcal{R}_0$ to (3.2), we formaly expand $\tilde{X}(\zeta)$ and $W(\zeta)$ with a parameter σ as

$$\tilde{X}(\zeta) = \sum_{n=1}^\infty \sigma^n \tilde{X}_n(\zeta), \quad W(\zeta) = \sum_{n=0}^\infty \sigma^{n+1} W_n(\zeta).$$

Substituting these into (3.2), we have for each order of σ

$$\begin{aligned}
A\tilde{X}_1 + 2aa_0 * \tilde{X}_1 &= W_0,\\
A\tilde{X}_2 + 2aa_0 * \tilde{X}_2 &= -a(\tilde{X}_1 * \tilde{X}_1) = W_1,\\
A\tilde{X}_3 + 2aa_0 * \tilde{X}_3 &= -a(\tilde{X}_1 * \tilde{X}_2 + \tilde{X}_2 * \tilde{X}_1) = W_2,\\
&\cdots\\
A\tilde{X}_{n+1} + 2aa_0 * \tilde{X}_{n+1} &= -a(\tilde{X}_1 * \tilde{X}_n + \tilde{X}_2 * \tilde{X}_{n-1} + \cdots + \tilde{X}_{n-1} * \tilde{X}_2 + \tilde{X}_n * \tilde{X}_1)\\
&= W_n,\\
&\cdots,
\end{aligned}$$

and, in the same way as above, each $\tilde{X}_n$ is given by

$$\tilde{X}_n = A^{-1}F_0 \int_0^\zeta \frac{W'_{n-1}}{F_0}\mathrm{d}\zeta'. \tag{3.6}$$

Let $L > 0$ be given arbitrarily. Then we can find $M > 0$ such that for $\zeta \in \mathcal{R}_\epsilon$ with $|\mathrm{Re}\ \zeta|, |\mathrm{Im}\ \zeta| \le L$

$$|\tilde{X}_1| < M|\zeta|. \tag{3.7}$$

Futhermore, there is $N > 0$ such that the derivative of $\tilde{X}_1$

$$\tilde{X}_1' = (A^{-1})'F_0 \int_0^\zeta \frac{W_0'}{F_0} \mathrm{d}\zeta' + A^{-1}F_0' \int_0^\zeta \frac{W_0'}{F_0} \mathrm{d}\zeta' + A^{-1}W_0' \tag{3.8}$$

satisfies the estimate

$$|\tilde{X}_1'| \le N(|\zeta| + 1). \tag{3.9}$$

for all $\zeta \in \mathcal{R}_\epsilon$ with $|\mathrm{Re}\ \zeta|, |\mathrm{Im}\ \zeta| \le L$.

Lemma 3.1. *Let $\zeta \in \mathcal{R}_\epsilon$ satisfy $|\mathrm{Re}\ \zeta|, |\mathrm{Im}\ \zeta| \le L$. Then for $n \ge 0$ the function $\tilde{X}_{n+1}$ is estimated as follows:*

$$|\tilde{X}_{n+1}(\zeta)| \le 2^n n! |a|^n M^{n+1} N^n \sum_{k=2n+1}^{3n+1} {}_nC_{k-2n-1} \frac{|\zeta|^k}{k!},$$
$$|\tilde{X}_{n+1}'(\zeta)| \le 2^n n! |a|^n M^n N^{n+1} \sum_{k=2n}^{3n+1} {}_{n+1}C_{k-2n} \frac{|\zeta|^k}{k!}.$$

Proof. We see from (3.7) and (3,9) that the inequalities are true for $n = 0$. Let $n \ge 0$, and suppose that the inequalities are true for n. Then, applying the following estimate for W_{n+1}' to (3.6)

$$\begin{aligned}
|W_{n+1}'| &= 2|a| \left| \tilde{X}_1' * \tilde{X}_{n+1} + \tilde{X}_2' * \tilde{X}_n + \cdots + \tilde{X}_{n+1}' * \tilde{X}_1 \right| \\
&\le 2|a| \left(|\tilde{X}_1' * \tilde{X}_{n+1}| + |\tilde{X}_2' * \tilde{X}_n| + \cdots + |\tilde{X}_{n+1}' * \tilde{X}_1| \right) \\
&\le 2^{n+1}(n+1)! |a|^{n+1} M^{n+1} N^{n+1} \sum_{k=2n+2}^{3n+3} {}_{n+1}C_{k-2n-2} \frac{|\zeta|^k}{k!},
\end{aligned}$$

we obtain

$$|\tilde{X}_{n+2}| \le 2^{n+1}(n+1)! |a|^{n+1} M^{n+2} N^{n+1} \sum_{k=2n+3}^{3n+4} {}_{n+1}C_{k-2n-3} \frac{|\zeta|^k}{k!},$$

and from the analogous formula to (3.8) it follows that the derivative of $\tilde{X}_{n+2}$ satisfies

$$|\tilde{X}_{n+2}'| \le 2^{n+1}(n+1)! |a|^{n+1} M^{n+1} N^{n+2} \sum_{k=2n+2}^{3n+4} {}_{n+1}C_{k-2n-3} \frac{|\zeta|^k}{k!},$$

Thus, we see that the inequalities are also true for $n+1$, and the lemma is obtained.

By using Lemma 3.1 and letting $\sigma = 1$ in $\tilde{X}$, X can be estimated as

$$\begin{aligned} |X| &= |a_0 + \tilde{X}_1 + \dots \tilde{X}_{n+1} + \dots| \\ &\le |a_0| + |\tilde{X}_1| + \dots |\tilde{X}_{n+1}| + \dots \\ &\le |a_0| + \dots + 2^{\frac{3n}{2}} \left[\frac{n}{2}\right]! \, |a|^{[\frac{n}{2}]} M^{[\frac{n}{2}]+1} N^{[\frac{n}{2}]} \frac{|\zeta|^n}{(n-1)!} + \dots . \end{aligned}$$

Since

$$\left\{ \frac{2^{\frac{3n}{2}} \left[\frac{n}{2}\right]! \, |a|^{[\frac{n}{2}]} M^{[\frac{n}{2}]+1} N^{[\frac{n}{2}]} |\zeta|^n}{(n-1)!} \right\}^{\frac{1}{n}} \to 0 \quad (n \to \infty),$$

$X(\zeta) = a_0 + \sum_{n=1}^{\infty} \tilde{X}_n(\zeta)$ uniformly converges on any bounded region of $\mathcal{R}_\epsilon$, which implies that $X(\zeta)$ is an analytic function on $\mathcal{R}_0$.

If $\beta = \pm 1$, then depending on the sign of β, we choose the region

$$\mathcal{R}_\epsilon^+ = \mathbb{C} \setminus \{s\zeta \mid s \in [1, +\infty), \quad \zeta \in D(\zeta_k^+, \epsilon), \quad k \in \mathbb{Z}\}$$

or

$$\mathcal{R}_\epsilon^- = \mathbb{C} \setminus \{s\zeta \mid s \in [1, +\infty), \quad \zeta \in D(\zeta_k^-, \epsilon), \quad k \in \mathbb{Z}\},$$

and apply the same algorithm as above in order to construct the solution $\tilde{X}(\zeta)$. Note that each $\tilde{X}_n$ is not singular at the points ζ_k^-'s if $\beta = +1$, and at points ζ_k^+'s if $\beta = -1$. In these cases, it is concluded that the the solution $X(\zeta) = a_0 + \tilde{X}(\zeta)$ to (3.2) is an analytic function on the region

$$\mathcal{R}_0^+ = \mathbb{C} \setminus \{s\zeta \mid s \in [1, +\infty), \zeta = \zeta_k^+, k \in \mathbb{Z}\}$$

if $\beta = +1$, and analytic on the region

$$\mathcal{R}_0^- = \mathbb{C} \setminus \{s\zeta \mid s \in [1, +\infty), \zeta = \zeta_k^-, k \in \mathbb{Z}\}$$

if $\beta = -1$. In the case where $\beta = 0, \pm 2, \pm 3, \cdots$, we obtain that $X(\zeta)$ has no singularities on $\mathbb{C}_\zeta$, i.e. an entire function.

We have chosen the first term a_0 to be arbitrary and applied the iteration algorithm to solve the functional equation (3.2) on a neighborhood of the origin $\zeta = 0$. The result is summarized as follows:

Proposition 3.2.

(1) *If β is not an integer, then the solution to the functional equation (3.2)*

$$X(\zeta) = a_0 + \sum_{n=1}^{\infty} \tilde{X}_n(\zeta)$$

uniformly converges on any compact subset of the region $\mathcal{R}_0$, and is an analytic function on $\mathcal{R}_0$.

(2) *If* $\beta = +1$, *then* $X(\zeta)$ *uniformly converges on any compact subset of the region* $\mathcal{R}_0^+$, *and is an analytic function on* $\mathcal{R}_0^+$.
(3) *If* $\beta = -1$, *then* $X(\zeta)$ *uniformly converges on any compact subset of the region* $\mathcal{R}_0^-$, *and is an analytic function on* $\mathcal{R}_0^-$.
(4) *If* $\beta = 0, \pm 2, \pm 3, \cdots$, *then* $X(\zeta)$ *is an entire function.*

Let us present here the concrete form of $X(\zeta)$ in a neighbourhood of the origin. We expand $A(\zeta)$, $F_0(\zeta)$ and $W_0'(\zeta)$ in terms of the Taylor series as

$$
\begin{aligned}
A(\zeta) &= (1-b-\lambda) - (1+b)\sum_{n=1}^{\infty} \frac{\zeta^{2n-1}}{(2n-1)!} + (1-b)\sum_{n=1}^{\infty} \frac{\zeta^{2n}}{(2n)!},\\
F_0(\zeta) &= \left(\frac{1-\alpha_1}{1-\alpha_2}\right)^{\beta} - \frac{\beta\zeta}{1-\alpha_1}\left(\frac{1-\alpha_1}{1-\alpha_2}\right)^{\beta}\left(1+\frac{1-\alpha_1}{1-\alpha_2}\right) + \dots,\\
W_0'(\zeta) &= (1+b)a_0 - aa_0^2 + a_0\left[-(1-b)\sum_{n=1}^{\infty}\frac{\zeta^{2n-1}}{(2n-1)!} + (1+b)\sum_{n=1}^{\infty}\frac{\zeta^{2n}}{(2n)!}\right].
\end{aligned}
$$

and substitute the above series into (3.6). Then iteration algorithm with the convolution in (3.6) that

$$
\tilde{X}_1 \mapsto W_1 \mapsto \tilde{X}_2 \mapsto W_2 \mapsto \cdots \mapsto W_{m-1} \mapsto \tilde{X}_m \mapsto \dots
$$

give rise to

$$
\begin{aligned}
\tilde{X}_1 &= a_{11}\zeta + a_{12}\zeta^2 + a_{13}\zeta^3 \dots,\\
\tilde{X}_2 &= a_{23}\zeta^3 + a_{24}\zeta^4 + a_{25}\zeta^5 \dots,\\
\tilde{X}_3 &= a_{35}\zeta^5 + a_{36}\zeta^6 + a_{37}\zeta^7 \dots,\\
&\dots\\
\tilde{X}_m &= a_{m2m-1}\zeta^{2m-1} + a_{m2m}\zeta^{2m} + a_{m2m+1}\zeta^{2m+1} \dots,\\
&\dots.
\end{aligned}
$$

It is remarkable that the coefficient a_{mn}'s are uniquely determined if the first term a_0 in $X(\zeta)$ is given. Thus, we have

$$
X(\zeta) = a_0 + \tilde{X}_1 + \tilde{X}_2 + \cdots = \sum_{n=0}^{\infty} a_n \zeta^n \tag{3.10}
$$

where the coefficient $a_n \in \mathbb{C}$ is given as

$$
a_n = a_{1n} + a_{2n} + a_{3n} + \cdots + a_{mn}, \quad (m \le n).
$$

The concrete forms of a_n's are given with computer assist as follows:

$$
\begin{aligned}
a_1 &= \frac{(1+b)a_0 - aa_0^2}{1-b-\lambda},\\
a_2 &= \frac{-1}{2(1-b-\lambda)}\left[\beta\left((1+b)a_0 - aa_0^2\right)\left(1-\frac{1-\alpha_1}{1-\alpha_2}\right) + (1-b)a_0\right]\\
&\quad - \frac{\left((1+b)a_0 - aa_0^2\right)}{(1-b-\lambda)}\left(\frac{1-\alpha_1}{1-\alpha_2}\right)^{\beta}\left[\frac{\beta}{1-\alpha_1}\left(1-\frac{1-\alpha_1}{1-\alpha_2}\right) - \frac{1+b}{1-b-\lambda}\right],\\
&\dots.
\end{aligned}
$$

§4 Analytic continuation of the local solutions

In this section we carry out the analytic continuation of the local solution $X(\zeta)$ constructed in §3 from a neighbourhood of the origin to the points $\zeta_k^{\pm}$, and show that the constant term a_0 in (3.1) and the index β in (3.4) are determined in considering the form of the function $X(\zeta)$ on a neighbourhood of $\zeta_k^{\pm}$.

In the case where $\beta = 0, \pm 2, \pm 3, \cdots$, from Propositin 3.2 it follows that $X(\zeta)$ is an entire funcitin, and hence the Laplace transform (2.2) gives a solution $x(t) = 0$, which is the trivial one. Thus, we can consider β to be not an integer or to be $\beta = \pm 1$.

Theorem 4.1. *If the Laplace transform (2.2) gives the non-trivial solution to the difference equation (2.1), then*

$$a_0 = \frac{\alpha + b\alpha^{-1}}{2a},$$

where $\alpha = \alpha_1, \alpha_2$ *and* $\alpha \neq 0$, *and* $\beta = +1$ *if* $\alpha = \alpha_1$ *and* $\beta = -1$ *if* $\alpha = \alpha_2$.

Proof. We suppose that β is not an integer, and derive a contradiction. By Proposition 3.2 there is an analytic function $X(\zeta)$ on $\mathcal{R}_0$ that is the solution on a neighbourhood of the origin $\zeta = 0$ to (2.3). Take and fix $k \in \mathbb{Z}$. In a neighbourhood of $\zeta_k^+ = \rho_+ + (2k\pi + \theta_+)\mathrm{i}$, the functions $A(\zeta)$, $F_0(\zeta)$ and $W_0'(\zeta)$ are expanded as

$$(4.1)\qquad \begin{aligned} A(\zeta) &= -A_{odd} \sum_{n=1}^{\infty} \frac{(\zeta - \zeta_k)^{2n-1}}{(2n-1)!} + A_{even} \sum_{n=1}^{\infty} \frac{(\zeta - \zeta_k)^{2n}}{(2n)!}, \\ F_0(\zeta) &= \left(\frac{\alpha_1}{\alpha_2 - \alpha_1}\right)^{\beta} (\zeta - \zeta_k)^{\beta} \left[1 + \mathcal{O}(\zeta - \zeta_k)\right], \\ W_0'(\zeta) &= -aa_0^2 + a_0 \left[A_{odd} \sum_{n=0}^{\infty} \frac{(\zeta - \zeta_k)^{2n}}{(2n)!} - A_{even} \sum_{n=1}^{\infty} \frac{(\zeta - \zeta_k)^{2n-1}}{(2n-1)!}\right], \end{aligned}$$

where $A_{odd} = \alpha + b\alpha^{-1}$ and $A_{even} = \alpha - b\alpha^{-1}$.

Now we express the variable ζ in (3.6) as

$$\zeta = \zeta_k + \xi.$$

For $\varepsilon > 0$ given small, let

$$|(1+\varepsilon)\zeta_k - (\zeta_k + \xi)| = |\varepsilon\zeta_k - \xi| \ll 1$$

and divide the integral into two parts as follows:

$$\int_0^{\zeta} = \int_0^{(1-\varepsilon)\zeta_k} + \int_{(1-\varepsilon)\zeta_k}^{\zeta_k + \xi}.$$

We introduce a *microfunction* $\arg \xi^{\beta}$ defined by

$$(4.2)\qquad \arg \xi^{\beta} = \xi^{\beta - 1} \left(\int_{-\varepsilon\zeta_k}^{\xi} \frac{\mathrm{d}\xi'}{\xi'^{\beta}} - \frac{1}{(\beta - 1)(-\varepsilon\zeta_k)^{\beta - 1}} \right).$$

Note that the argument of the function $\xi^{\beta-1}$ outside the integral in (4.2) is determined, while the integration path in the integral is not yet determined in this stage. Under the above decomposition of the integral together with the expansions (4.1), the integral (3.6) with $n=1$ is expressed as

$$\begin{aligned}\tilde{X}_1 &= A^{-1}F_0 \int_0^{\zeta} \frac{W_0'}{F_0} d\zeta' \\ &= \arg \xi^{\beta} \, (b_{10}' + b_{11}'\xi + b_{12}'\xi^2 + \dots) + R_1(\xi)\end{aligned} \tag{4.3}$$

where $b_{1n}' \in \mathbb{C}$, $n = 0,1,2,\cdots$ and $R_1(\xi)$ is a regular function of ξ.

The iteration algorithm stated in §3 gives a series of the functions:

$$\begin{aligned}W_1 &= -a\tilde{X}_1 * \tilde{X}_1 \\ &= \arg \xi^{\beta} \, (v_{12}\xi^2 + v_{13}\xi^3 + \dots) + r_1(\xi), \\ \tilde{X}_2 &= A^{-1}F_0 \int_0^{\zeta} \frac{W_1'}{F_0} d\zeta' \\ &= \arg \xi^{\beta} \, (b_{21}'\xi + b_{22}'\xi^2 + \dots) + R_2(\xi), \\ W_2 &= -2a\tilde{X}_1 * \tilde{X}_2 \\ &= \arg \xi^{\beta} \, (v_{23}\xi^3 + v_{24}\xi^4 + \dots) + r_2(\xi), \\ &\dots\end{aligned} \tag{4.4}$$

where b_{mn}', $v_{mn} \in \mathbb{C}$, $m,n = 1,2,\cdots$, and $R_m(\xi)$ and $r_m(\xi)$ are regular functions of ξ. Here we used the following relation to caluculate the convolution integral in W_m:

$$\begin{aligned}&\int_0^{\zeta} \tilde{X}_m(\zeta-\zeta')\tilde{X}_n(\zeta')d\zeta' \\ &= 2\int_{\frac{\zeta}{2}}^{\zeta} \tilde{X}_m(\zeta-\zeta')\tilde{X}_n(\zeta')d\zeta' \\ &= 2\int_{\frac{\zeta}{2}}^{(1-\varepsilon)\zeta_k} \tilde{X}_m(\zeta-\zeta')\tilde{X}_n(\zeta')d\zeta' + 2\int_{(1-\varepsilon)\zeta_k}^{\zeta} \tilde{X}_m(\zeta-\zeta')\tilde{X}_n(\zeta')d\zeta'\end{aligned}$$

where the first integral only gives a regular function, while the second one contributes to the part of the microfunction in terms of the relation $\zeta = \zeta_k + \xi$. From (4.3) and (4.4), it turns out that the function $X(\zeta) = a_0 + \tilde{X}(\zeta)$ in a neighbourhood of ζ_k is given as

$$X(\zeta) = \arg \xi^{\beta} \sum_{n=0}^{\infty} b_n(\zeta-\zeta_k)^n + R(\zeta-\zeta_k) \tag{4.5}$$

where $b_n \in \mathbb{C}$, $n = 0,1,2,\cdots$ and $R(\zeta-\zeta_k)$ is a regular function in a neighbourhood of $\zeta = \zeta_k$. Note that the function $X(\zeta)$ has the same form and the each value of b_n does not depend on the choice of $\zeta = \zeta_k^+$ $(k \in \mathbb{Z})$. To obtain the concrete form of the coefficients b_n's, we need to solve the functional equation (3.2) in terms of the function

$X(\zeta)$ given by (4.5). To do this, we use the "var" operator introduced by Écalle [E], which is used to caluculate the Laplace transform of microfunctions and defined as

$$\mathrm{var}F(\zeta) = F(\zeta e^{2\pi i}) - F(\zeta) \tag{4.7}$$

for a microfunction $F(\zeta)$. We apply the variational operator var to the functional equation (3.2) to obtain a univalued function. Taking var of the function $X(\zeta)$, we have

$$\mathrm{var}\ X(\zeta) = \left(e^{2\pi i\beta_r}e^{-2\pi\beta_i} - 1\right)\sum_{n=0}^{\infty} b_n(\zeta-\zeta_k)^n \tag{4.7}$$

where $\beta = \beta_r + i\beta_i$ and $\beta_r, \beta_i \in \mathbb{R}$. Note that for the regular function $R(\zeta)$, var $R(\zeta) = 0$.

Taking var on both sides of (2.3) and substituting (4.4) into that, we have

$$\begin{aligned}&\left(-A_{odd}\xi + \frac{A_{even}}{2}\xi^2 - \dots\right)(b_0 + b_1\xi + \dots)\\ &= -2a\int_0^{\xi}(a_0 + a_1(\xi-\xi') + \dots)(b_0 + b_1\xi' + \dots)d\xi'.\end{aligned}$$

The first order $\mathcal{O}(\xi)$ gives the following relation for a_0:

$$a_0 = \frac{A_{odd}}{2a}. \tag{4.8}$$

Substituting (4.8) into the definition of β in (3.4) and taking account of the fact that α_1 and α_2 are two solutions to the quadratic equation $\zeta^2 - \lambda\zeta - b = 0$, we obtain

$$\beta = \frac{2aa_0}{\alpha_1-\alpha_2} = \frac{A_{odd}}{\alpha_1-\alpha_2} = \begin{cases}\frac{\alpha_1+b\alpha_1^{-1}}{2\alpha_1-\lambda} = 1 & \text{if } \alpha=\alpha_1\\ \frac{\alpha_2+b\alpha_2^{-1}}{-2\alpha_2+\lambda} = -1 & \text{if } \alpha=\alpha_2,\end{cases}$$

which contradicts the assumption that β is not an integer. Therefore, the theorem is obtained.

From Theorem 4.1 it follows that $\beta = \pm 1$. If $\beta = 1$, then by Proposition 3.2 the solution $X(\zeta)$ on a neighbourhood of the origin to (2.3) is given as an analytic function on the region $\mathcal{R}_0^+$. In the case of $\beta = -1$ the solution $X(\zeta)$ is given as that on the region $\mathcal{R}_0^-$. In this stage, it is not necessary to distinguish between $\beta = +1$ and $\beta = -1$. Thus, in the following, $\mathcal{R}_0^+$ and $\mathcal{R}_0^-$ are denoted by the same symbol $\mathcal{R}_0$, and ζ_k^+ and ζ_k^- are denoted by the same symbol ζ_k.

Theorem 4.2. *The form of the function $X(\zeta)$ in a neighbourhood of each singularity $\zeta = \zeta_k$ is given by*

$$X(\zeta) = \sum_{n=0}^{\infty} b_n(\zeta-\zeta_k)^n\log(\zeta-\zeta_k) + R(\zeta-\zeta_k) \tag{4.9}$$

with complex coefficents b_{mn} and a regular function $\tilde{R}(\zeta)$.

Proof. The analogous iteration algorithm used in the proof of Theorem 4.1 give rise to

$$
\begin{aligned}
\tilde{X}_1 &= (\tilde{b}_{10} + \tilde{b}_{11}\xi + \dots) \log \xi + \tilde{R}_1(\xi), \\
W_1 &= (\tilde{v}_{11}\xi + \tilde{v}_{12}\xi^2 + \dots) \log \xi + \tilde{r}_1(\xi), \\
\tilde{X}_2 &= (\tilde{b}_{21}\xi + +\tilde{b}_{22}\xi^2 + \dots) \log \xi + \tilde{R}_2(\xi), \\
W_2 &= (\tilde{v}_{22}\xi^2 + \tilde{v}_{23}\xi^3 + \dots) \log \xi + \tilde{r}_2(\xi), \\
&\dots \\
\tilde{X}_m &= (\tilde{b}_{mm-1}\xi^{m-1} + +\tilde{b}_{mm}\xi^m + \dots) \log \xi + \tilde{R}_m(\xi), \\
W_m &= (\tilde{v}_{mm}\xi^m + \tilde{v}_{mm+1}\xi^{m+1} + \dots) \log \xi + \tilde{r}_m(\xi), \\
&\dots
\end{aligned}
$$

where the coefficients $\tilde{b}_{mn}$ and $\tilde{v}_{mn}$ are complex numbers, while $\tilde{R}_m(\xi)$ and $\tilde{r}_m(\xi)$ are regular functions of ξ. This sequence of functions gives the function $X(\zeta)$ of form (4.9).

§5 Global solution to the integral equation

In this section we give the solution X to the integral equation (2.3).

As before, let $\alpha \neq 0$ be one of eigenvalues of the derivative Df_P at P. We define the lattice Γ_α generated by $-\log \alpha$ as follows. For $k \in \mathbb{Z}$ let $\zeta_k = \rho + (2k\pi + \theta)\mathrm{i}$, where

$$\rho = -\log|\alpha|, \quad -\pi < \theta = \arg \alpha \leq \pi,$$

and let

$$\Gamma_\alpha = \{\zeta \in \mathbb{C} \mid \zeta = \sum_{\ell=1}^{N} \zeta_{k_\ell},\ N = 1, 2, \cdots\}.$$

It is easy to see that Γ_α is on the right half plane of $\mathbb{C}$ if $0 < |\alpha| < 1$, on the left half plane if $|\alpha| > 1$, and on the imaginary axis if $|\alpha| = 1$. Note that Γ_α is dense in the imaginary axis in the case of $|\alpha| = 1$.

Lemma 5.1. *Let $|\alpha| \neq 1$. For $\zeta \in \mathbb{C} \setminus \Gamma_\alpha$ and a path ω from the origin to ζ in $\mathbb{C} \setminus \Gamma_\alpha$, there is a smooth path δ from the origin to ζ homotopic to ω in $\mathbb{C} \setminus \Gamma_\alpha$ such that $\zeta/2 \in \delta$ and δ is symmetrical with respect to $\zeta/2$.*

By Lemma 5.1 we can perform the following algorithm;

$$X_{n+1}^{(N)} = A^{-1} F_0 \int_\delta \frac{W_n'}{F_0} \mathrm{d}\zeta',$$

$$W_n' = -2a(X_n^{(N)} * X_1^{(N)\prime} + \cdots + X_1 * X_n^{(N)\prime}).$$

Then

$$X^{(N)} = X_1^{(N)} + X_2^{(N)} + \cdots + X_n^{(N)} + \cdots$$

is a Riemann suface, and

$$X = \lim_{N \to \infty} X^{(N)}$$

is the solution to the integral equation (2.3). In the case where $|\alpha| = 1$, we can perform the algorithm above and obtain the solution.

Theorem 5.2. *Let* $\zeta == \sum_{\ell=1}^{N} \zeta_{k_\ell} + \xi$, *and let* ω *be a path from the origin to* ζ *in* $\mathbb{C} \setminus \Gamma_\alpha$. *Then* $X(\zeta) = X(\zeta, \omega)$ *is given by the sum* $\sum_{n=1}^{\infty} \tilde{X}_n$ *of the following functions;*

$$\begin{aligned}
\tilde{X}_1 &= \mathrm{reg}(\xi), \\
\tilde{X}_2 &= (*\xi + *\xi^2 + *\xi^3 + \cdots) \log \xi + \mathrm{reg}(\xi), \\
\tilde{X}_3 &= (*\xi^2 + *\xi^3 + *\xi^4 + \cdots)(\log \xi)^2 \\
&\quad + (*\xi^2 + *\xi^3 + *\xi^4 + \cdots) \log \xi + \mathrm{reg}(\xi), \\
\tilde{X}_4 &= (*\xi^3 + *\xi^4 + *\xi^5 + \cdots)(\log \xi)^3 \\
&\quad + (*\xi^3 + *\xi^4 + *\xi^5 + \cdots)(\log \xi)^2 \\
&\quad + (*\xi^3 + *\xi^4 + *\xi^5 + \cdots) \log \xi + \mathrm{reg}(\xi), \\
&\cdots \\
\tilde{X}_{N-1} &= (*\xi^{N-2} + *\xi^{N-1} + *\xi^N + \cdots)(\log \xi)^{N-2} \\
&\quad + \cdots \\
&\quad + (*\xi^{N-2} + *\xi^{N-1} + *\xi^N + \cdots) \log \xi + \mathrm{reg}(\xi), \\
\tilde{X}_N &= (*\xi^{N-1} + *\xi^N + *\xi^{N+1} + \cdots)(\log \xi)^N \\
&\quad + \cdots \\
&\quad + (*\xi^{N-1} + *\xi^N + *\xi^{N+1} + \cdots) \log \xi + \mathrm{reg}(\xi), \\
\tilde{X}_{N+1} &= (*\xi^N + *\xi^{N+1} + *\xi^{N+2} + \cdots)(\log \xi)^N \\
&\quad + \cdots \\
&\quad + (*\xi^N + *\xi^{N+1} + *\xi^{N+2} + \cdots) \log \xi + \mathrm{reg}(\xi), \\
&\cdots .
\end{aligned}$$

where $*$ *'s are complex coefficients and* $\mathrm{reg}(\xi)$ *'s are regular functions of* ξ.

We remark that the complex coefficients $*$'s are written by a, b, a_0, α and the special values of the Hurwitz zeta, and that if $a, b, \alpha \in \mathbb{R}$ then all coefficients $*$'s are also real numbers.

§6 Resurgent functions and Laplace transformations

In the case where $|\alpha| \neq 1$, we can obtain the *resurgent functions* X_R from the Riemann surface X along the each line L_k connecting the origin and ζ_k. Then we define

$$x(t) = \int_0^{e^{i\Theta_k}\infty} e^{-\zeta t} X_R(\zeta) d\zeta,$$

where Θ_k is the angle of L_k. It can be proved that $x(t)$ is an analytic function defined on the whole plane $\mathbb{C}$, and that $x(t)$ does not depend on the choice of ζ_k. If we let

$$\varphi(t) = \begin{pmatrix} x(t) + x_f \\ bx(t-1) + y_f \end{pmatrix},$$

then $\varphi : \mathbb{C} \to \mathbb{C}^2$ satisfies the functional equation (0.1) and is different from the Poincaré maps in the sense mentioned before.

The case of $|\alpha| = 1$ is also discussed in the similar manner, and Theorem 1 can be proved.

For the details of this paper, the authors hope to appear elsewhere.

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